*Journal of Statistical Physics, Vol. 112, Nos. 1/2, July 2003 (© 2003)*

# The Riemann Surface of the Chiral Potts Model Free Energy Function

**R. J. Baxter**[1]

*Received December 3, 2002; accepted February 4, 2003*

In a recent paper we derived the free energy or partition function of the $N$-state chiral Potts model by using the infinite lattice "inversion relation" method, together with a non-obvious extra symmetry. This gave us three recursion relations for the partition function per site $T_{pq}$ of the infinite lattice. Here we use these recursion relations to obtain the full Riemann surface of $T_{pq}$. In terms of the $t_p$, $t_q$ variables, it consists of an infinite number of Riemann sheets, each sheet corresponding to a point on a $(2N-1)$-dimensional lattice (for $N>2$). The function $T_{pq}$ is meromorphic on this surface: we obtain the orders of all the zeros and poles. For $N$ odd, we show that these orders are determined by the usual inversion and rotation relations (without the extra symmetry), together with a simple linearity ansatz. For $N$ even, this method does not give the orders uniquely, but leaves only $[(N+4)/4]$ parameters to be determined.

**KEY WORDS:** Statistical mechanics; lattice models; chiral Potts model; free energy.

## 1. INTRODUCTION

The free energy of the $N$-state chiral Potts model was obtained implicitly in 1988[1] from the star-triangle relation, and explicitly in 1990[2,3] from the functional relations for the finite-lattice transfer matrices. (The two methods have since been shown to be equivalent.)[4,5] Recently[6] we considered the infinite-lattice limit of these functional relations, and showed

[1] Theoretical Physics, I.A.S. and School of Mathematical Sciences, The Australian National University, Canberra, Australian Capital Territory 0200, Australia.

that they reduced to the rotation and inversion relations (which exist for most two-dimensional models, solvable or not),[7] or rather to a single combined rotation-inversion relation. All other equations were either definitions of the auxiliary function $\tau_2(t_q)$, or direct consequences of the relations and definitions, so by themselves contained no new information.

We showed that the reason we could solve the infinite relations was that we knew the analyticity properties in some "central" domain that included part of the physical regime. In particular, we knew that $\tau_2(t_q)$ had only one branch cut in the $t_q$-plane, instead of the $N$ cuts one might expect. This was equivalent to an extra symmetry for the free energy. It ensured that one could use Wiener–Hopf methods to calculate the partition function per site $T_{pq}$.

We found that $T_{pq}$ satisfied three "recursion" relations: Eqs. (70), (71), and (75) of ref. 6. (We could have obtained these directly from the solution found in 1990.) Here we use these three relations to obtain the full Riemann surface (in both the $p$ and the $q$ rapidity variables) on which $T_{pq}$ lives. It is a meromorphic function on this surface, its only singularities being poles. Its poles and zeros occur only when $t_p^N = t_q^N$, $\lambda_p = \lambda_q^{\pm 1}$.

Regarding $T_{pq}$ as a function of $t_p$, $t_q$, its Riemann surface consists of an infinite number of Riemann sheets. Each sheet is specified by a set of $N$ integers $m_0,..., m_{N-1}$ associated with the $p$-variable, and another set $n_0,..., n_{N-1}$ associated with the $q$-variable. Thus it corresponds to a point on a $2N$-dimensional lattice. In fact, although the chiral Potts model does not have the usual difference property (that one can choose the rapidities $p, q$ so that the Boltzmann weights, correlations and free energy depend only on $p, q$ via their difference $p-q$), we do find a weak residual version of this property: $T_{pq}$ is the same on all sheets obtained by incrementing $m_0,..., m_{N-1}, n_0,..., n_{N-1}$ by an arbitrary integer. Hence the Riemann surface can be associated with a lattice of reduced dimension $2N-1$.

The case $N=2$, when the model reduces to the Ising model, is special. Then two of the integers $m_0,..., n_{N-1}$ vanish, so the surface is one-dimensional, being that of the elliptic function argument $z = \exp[\pi(u_p - u_q)/K']$ of Eq. (B7) of ref. 6.

A knowledge of the Riemann surface, and of the orders of the poles and zeros, gives us a great deal of information about the function $T_{pq}$. We observe that these orders are linear in the $m_0,..., m_{N-1}$, and in the $n_0,..., n_{N-1}$. Hence they are bilinear in the full set of $2N$ integers $m_0,..., m_{N-1}, n_0,..., n_{N-1}$, depending only on their differences.

We ask the question: is this bilinearity, together with the basic inversion and rotation relations sufficient to determine the orders of the poles and zeros? We find that for $N$ odd the answer is yes. For $N$ even it is not quite sufficient: there are $[(N+4)/4]$ parameters still undetermined.

## 2. DOMAINS AND RIEMANN SHEETS

The Boltzmann weights of the $N$-state chiral Potts model are functions of a constant $k$, and two "rapidity" variables $p$ and $q$.[2,3] Let

$$k' = (1-k^2)^{1/2}, \qquad \omega = e^{2\pi i/N}. \tag{1}$$

Then the variable $p$ can be thought of as a point $(x_p, y_p, t_p, \lambda_p)$ on the algebraic curve

$$\begin{aligned} x_p^N + y_p^N &= k(1+x_p^N y_p^N), \qquad & t_p &= x_p y_p, \\ k x_p^N &= 1 - k'/\lambda_p, \qquad & k y_p^N &= 1 - k'\lambda_p. \end{aligned} \tag{2}$$

Similarly $q$ is the point $(x_q, y_q, t_q)$.

As we remark in ref. 6, if $x_p, x_q, y_p, y_q, \omega x_p$ all lie on the unit circle and are ordered anti-cyclically around it, then all the Boltzmann weights are real and positive, and therefore so is the partition function per site $T_{pq}$. We refer to this case as the *physical regime*. Outside this regime we define $T_{pq}$ by analytic continuation.

This function $T_{pq}$ therefore lives on a Riemann surface in both the $p$ and the $q$ variables. To specify this surface, first consider the $p$ variable. If $|\lambda_p| < 1$, then $x_p$ lies in the region $\mathscr{S}$ of Fig. 1, while $y_p$ lies in one of the $N$ approximately circular regions $\mathscr{R}_0, \ldots, \mathscr{R}_{N-1}$ surrounding the points $1, \omega, \ldots, \omega^{N-1}$.

These regions shrink to points in the low-temperature limit $k' \to 0$, so in this limit it is certainly true that

$$y_p \simeq \omega^r \qquad \text{if} \quad y_p \in \mathscr{R}_r.$$

We find it helpful to write "$y_p \simeq \omega^r$," by which we mean "$y_p \in \mathscr{R}_r$."

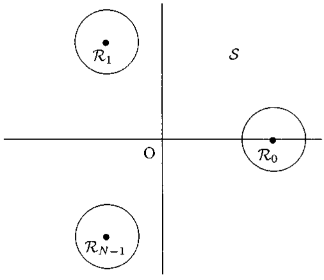


Fig. 1. The $N+1$ regions $\mathscr{S}, \mathscr{R}_0, \ldots, \mathscr{R}_{N-1}$ of the complex plane in which $x_q$ and $y_q$ lie (for $N=3$). $\mathscr{R}_0, \ldots, \mathscr{R}_{N-1}$ are the interiors of the approximate circles centred on $1, \omega, \ldots, \omega^{N-1}$. $\mathscr{S}$ is the complex plane outside all $N$ circles.

We define a "domain" $\mathscr{D}_r$ for $x_p$ and $y_p$ simultaneously by saying that if $y_p$ lies in $\mathscr{R}_r$, then $(x_p, y_p)$ lies in the domain $\mathscr{D}_r$. More simply, we say that $p$ lies in $\mathscr{D}_r$. We also say in this case that $p$ has *parity* 0 and that it has *type* $r$.

Conversely, if $|\lambda_p| > 1$, then $y_p$ lies in the region $\mathscr{S}$ and $x_p$ lies in one of $\mathscr{R}_0, \ldots, \mathscr{R}_{N-1}$. If $x_p$ lies in $\mathscr{R}_r$, then we say that $(x_p, y_p)$ or $p$ lies in the domain $\mathscr{D}'_r$, has parity 1 and type $r$. Let $\rho, r$ be the parity and type of $p$. Then

$$
\begin{aligned}
&x_p \in \mathscr{S}, \quad y_p \in \mathscr{R}_r, \quad |\lambda_p| < 1 \quad \text{and} \quad p \in \mathscr{D}_r \quad \text{if} \quad \rho = 0, \\
&x_p \in \mathscr{R}_r, \quad y_p \in \mathscr{S}, \quad |\lambda_p| > 1 \quad \text{and} \quad p \in \mathscr{D}'_r \quad \text{if} \quad \rho = 1.
\end{aligned} \tag{3}
$$

We refer to the case when $p$ and $q$ both lie in $\mathscr{D}_0$, so that $|\lambda_p| < 1$, $|\lambda_q| < 1$, $y_p, y_q \in \mathscr{R}_0$, $x_p, x_q \in \mathscr{S}$, as the *central* regime or domain. It overlaps the physical regime, so $T_{pq}$ is readily extended to this domain. Series expansions strongly suggest that it has no poles or zeros in the central domain (as can be verified from the explicit solution (67) of ref. 6).

If we analytically continue from the central domain to $|\lambda_p| > 1$, then as $\lambda_p$ crosses the unit circle, $x_p$ enters one of the regions $\mathscr{R}_0, \ldots, \mathscr{R}_{N-1}$, say $\mathscr{R}_{r_1}$, while $y_p$ leaves $\mathscr{R}_0$ and enters $\mathscr{S}$. Thus $p$ goes from the domain $\mathscr{D}_0$ to $\mathscr{D}'_{r_1}$. Its type changes from 0 to $r_1$, and its parity from 0 to 1.

We can then analytically continue from $|\lambda_p| > 1$ back to $|\lambda_p| < 1$, but $p$ will not necessarily return to $\mathscr{D}_0$: in general it will go to a new domain $\mathscr{D}_{r_2}$. And so on: $p$ will move successively through a sequence of domains

$$
\mathscr{D}_0, \quad \mathscr{D}'_{r_1}, \quad \mathscr{D}_{r_2}, \quad \mathscr{D}'_{r_3}, \quad \mathscr{D}_{r_4}, \ldots \tag{4}
$$

Their types are $0, r_1, r_2, r_3, r_4, \ldots$, and their parities are $0, 1, 0, 1, 0, \ldots$ .

From (2),

$$
k^2 t_p^N = 1 - k'(\lambda_p + 1/\lambda_p) + k'^2,
$$

so the unit circle in the $\lambda_p$-plane corresponds to $N$ straight-line segments in the $t_p$-plane, from $\omega^j \eta$ to $\omega^j/\eta$, as in Fig. 2 of ref. 6. Here $\eta = [(1-k')/(1+k')]^{1/N}$. In terms of the $t_p$ variable, each $(x_p, y_p)$ domain is therefore a Riemann sheet, with branch cuts along these straight-line segments. As one goes from a domain to a neighbouring domain, $t_p$ crosses one of these $N$ branch cuts and goes from one sheet to the next. We shall use the words "domain," "Riemann sheet" and "sheet" interchangeably.

We define $T_{pq}$ by analytic continuation as $p$ moves through this sequence, starting from some initial value in $\mathscr{D}_0$. As $p$ moves from one domain to the next, $T_{pq}$ moves from one Riemann sheet to the next.

At this stage we do not know the Riemann surface on which $T_{pq}$ lives (i.e., how the Riemann sheets connect with one another), so we must be prepared to remember this full sequence of domains in order to uniquely specify the value of $T_{pq}$ at a point $p$. Since the parities alternate, it is sufficient to remember the domain types $\{r_1, r_2, r_3, r_4,...\}$.

Similarly, if initially the other rapidity variable $q$ is in the central domain $\mathscr{D}_0$, and is analytically continued through the sequence

$$\mathscr{D}_0, \quad \mathscr{D}'_{s_1}, \quad \mathscr{D}_{s_2}, \quad \mathscr{D}'_{s_3}, \quad \mathscr{D}_{s_4},...$$

we must remember the sequence $\{s_1, s_2, s_3, s_4,...\}$.

Fortunately it seems that $T_{pq}$ does not depend on whether $p$ or $q$ moves first: i.e., how the two sequences $\{r_1, r_2, r_3, r_4,...\}$, $\{s_1, s_2, s_3, s_4,...\}$ are interleaved. Thus the Riemann surface in both variables is simply the union of the two single-variable surfaces.

## 3. RECURSION RELATIONS FOR $T_{pq}$

In ref. 6 we obtained the two relations, true for $r, s = 0,..., N-1$:

$$T_r(\omega^r y_p, x_p \mid x_q, y_q)$$

$$= \frac{N\, T(\omega^r x_p, y_p \mid x_q, y_q)}{T(\omega^{-1} x_p, y_p \mid x_q, y_q)\, T(x_p, y_p \mid x_q, y_q)} \prod_{j=1}^{r} \frac{t_p - \omega^{-j} t_q}{(x_p - \omega^{-j} x_q)(y_p - \omega^{-j} y_q)}$$

$$\times \prod_{j=r+1}^{N-1} \frac{t_p - \omega^{-j} t_q}{(x_p - \omega^{-j} y_q)(y_p - \omega^{-j} x_q)}, \tag{5}$$

$$T_s(x_p, y_p \mid \omega^s y_q, x_q)$$

$$= \frac{N\, T(x_p, y_p \mid \omega^s x_q, y_q)}{T(x_p, y_p \mid \omega^{-1} x_q, y_q)\, T(x_p, y_p \mid x_q, y_q)} \prod_{j=1}^{s} \frac{t_p - \omega^{j-1} t_q}{(x_p - \omega^{j-1} y_q)(y_p - \omega^{j} x_q)}$$

$$\times \prod_{j=s+1}^{N-1} \frac{t_p - \omega^{j-1} t_q}{(x_p - \omega^{j-1} x_q)(y_p - \omega^{j} y_q)}, \tag{6}$$

together with

$$\frac{T(x_p, y_p \mid x_q, y_q)\, T(\omega x_p, y_p \mid \omega x_q, y_q)}{T(x_p, y_p \mid \omega x_q, y_q)\, T(\omega x_p, y_p \mid x_q, y_q)} = \frac{(x_p - x_q)(\omega t_p - t_q)}{(\omega x_p - x_q)(t_p - t_q)}. \tag{7}$$

In these three relations the $p$-arguments $(\omega^i x_p, y_p)$ (for any $i$) of the function $T$ (without a suffix) lie in the initial domain $\mathscr{D}_0$, and so do the

$q$-arguments $(\omega^i x_q, y_q)$. The function $T_r$ on the lhs of (5) has $p$-arguments $(x'_p, y'_p) = (\omega^r y_p, x_p)$ lying in the domain $\mathscr{D}'_r$ adjacent to $\mathscr{D}_0$. Similarly, $T_s$ on the lhs of (6) has $q$-arguments $(x'_q, y'_q) = (\omega^s y_q, x_q)$ lying in the domain $\mathscr{D}'_s$ adjacent to $\mathscr{D}_0$.

We can use (5) and (6) to obtain the function $T$ on any of its infinitely many Riemann sheets. For instance, if in (5) we allow $p = (x_p, y_p)$ to leave $\mathscr{D}_0$ and enter the neighbouring domain $\mathscr{D}'_m$, where $x_p \simeq \omega^m$, then the arguments of $T$ on the lhs will move to a sheet $\mathscr{D}_{rm}$. The three $T$s on the rhs move to the domains $\mathscr{D}'_{m+r}$, $\mathscr{D}'_{m-1}$, $\mathscr{D}'_m$, respectively. These can in turn be expressed in terms of $T$ in $\mathscr{D}_0$ by again using (5).

We can repeat this procedure *ad infinitem*. At each stage the function $T_{pq}$ on the left-hand side of (5) moves to a new domain or sheet, while the ones on the rhs move to domains or sheets that have already been expressed in terms of the values of $T_{pq}$ in the initial domain $\mathscr{D}_0$. Similarly, we can use (6) to obtain $T_{pq}$ on successive sheets in the $q$-variable.

At first sight we will obtain an exponentially infinite Cayley tree, each $(p, q)$ surface having $2N$ neighbours ($N$ neighbours in the $p$ variable, $N$ in the $q$-variable). The number of $m$th neighbours of the initial domain (where $p, q \in \mathscr{D}_0$) will be $2N \times (2N-1)^{m-1}$.

However, it is not as as bad as that. Consider the function $T_{pq}$ on a Riemann sheet of $p$-parity $\rho$ and $p$-type $r$, and of $q$-parity $\sigma$ and $q$-type $s$. Thus $\rho, \sigma = 0, 1$ and $r, s = 0, \dots, N-1$. Define an associated function $A^{\rho\sigma}_{rs}(a_p, b_p \mid a_q, b_q)$ by

$$
\begin{aligned}
T(x_p, y_p \mid x_q, y_q) &= A^{00}_{rs}(x_p, \omega^{-r} y_p \mid x_q, \omega^{-s} y_q) && \text{if} \quad \rho = 0, \quad \sigma = 0, \\
&= N / A^{01}_{rs}(x_p, \omega^{-r} y_p \mid y_q, \omega^{-s} x_q) && \text{if} \quad \rho = 0, \quad \sigma = 1, \\
&= N / A^{10}_{rs}(y_p, \omega^{-r} x_p \mid x_q, \omega^{-s} y_q) && \text{if} \quad \rho = 1, \quad \sigma = 0, \\
&= A^{11}_{rs}(y_p, \omega^{-r} x_p \mid y_q, \omega^{-s} x_q) && \text{if} \quad \rho = 1, \quad \sigma = 1.
\end{aligned}
\tag{8}
$$

In each case the $p$-arguments $a_p, b_p$ of $A^{\rho\sigma}_{rs}(a_p, b_p \mid a_q, b_q)$ lie in $\mathscr{D}_0$, and so do the $q$-arguments $a_q, b_q$.

Then no matter how many times we iterate (5) and (6), the resulting function $T_{pq}$ will have an associated function of the form

$$
\begin{aligned}
&A^{\rho\sigma}_{rs}(a_p, b_p \mid a_q, b_q) \\
&\quad = \prod_{j=0}^{N-1} \left[ (a_p - \omega^j a_q)^{\alpha_j} (a_p - \omega^j b_q)^{\alpha'_j} (b_p - \omega^j a_q)^{\beta_j} (b_p - \omega^j b_q)^{\beta'_j} (a_p b_p - \omega^j a_q b_q)^{\gamma_j} \right] \\
&\quad \times \prod_{i=0}^{N-1} \prod_{j=0}^{N-1} T(\omega^i a_p, b_p \mid \omega^j a_q, b_q)^{\rho_{ij}},
\end{aligned}
\tag{9}
$$

where the exponents $\alpha_j, \ldots, \rho_{ij}$ are integers and the functions $T$ on the rhs all have $p$ and $q$ arguments lying in the initial sheet $\mathscr{D}_0$.

Further, we can always use (7) to eliminate the $T(\omega^i a_p, b_p \mid \omega^j a_q, b_q)$ with $i$ and $j$ greater than zero, so we can require that

$$\rho_{ij} = 0 \qquad \text{if} \quad i \qquad \text{and} \qquad j > 0, \tag{10}$$

leaving only $\rho_{00}$, $\rho_{i0}$ and $\rho_{0j}$ $(i, j > 0)$ as possibly non-zero. There are then at most $5N + 2N - 1 = 7N - 1$ non-zero exponents. Equating any two Riemann surfaces with the same values of the function $T_{pq}$, it follows that any Riemann surface (in both the $p$ and $q$ variables) can be specified by at most $7N - 1$ arbitrary integers (together with the two-valued integers $\rho, \sigma$ and the $N$-valued integers $r, s$), so can be regarded as corresponding to a point on a $(7N-1)$-dimensional lattice.

In fact the lattice is of lower dimension still, since there are many relations between the exponents. For any rational number $x$, let $[x]$ denote its integer part, so that

$$[x] \leqslant x < [x] + 1,$$

and define

$$F_{i,j} = \left[\frac{i-1}{N}\right] - \left[\frac{i-j-1}{N}\right] - \left[\frac{j}{N}\right]. \tag{11}$$

Then $F_{ij}$ has value 0 or 1, and is periodic in $i, j$ of period $N$, i.e., $F_{i+N,j} = F_{i,j+N} = F_{ij}$.

The relations between the exponents are conveniently expressed by associating with each Riemann sheet two more sets of integers:[2]

$$m = \{m_0, \ldots, m_{N-1}\}, \qquad n = \{n_0, \ldots, n_{N-1}\}.$$

The set $m$ is varied when the $p$-domain is changed, and $n$ when the $q$-domain is changed. If $m$ is the set on a sheet of parities $\rho, \sigma$ and types $r, s$, and $m'$ is the set $m$ on a $p$-neighbouring sheet of parities $\rho', \sigma$ and types $r', s$ (the $q$ variables being the same on each sheet), then $\rho' = 1 - \rho$ and

$$m'_j = m_{r+r'} + m_{r+r'+1} - m_j + \rho(1 - \delta_{r+r',0}) - F_{j,r+r'}, \tag{12}$$

[2] We originally found the $n_j$ by keeping the $p$ variables in $\mathscr{D}_0$ and looking at the orders of the zeros of $T_{pq}$, considered as a function of $q$. Then $m_j = 0$ and the rhs of (25) becomes $2n_{-k} + w_k$. In the normalization where the Boltzmann weights are free of poles we lose the $w_k$ and the zeros are simply $2(-1)^\sigma n_{-k}$. Similarly, one can obtain the $m_j$.

for $j = 0, \dots, N-1$, using the periodic convention

$$m_{j+N} = m_j, \qquad \forall j. \tag{13}$$

Similarly, the set $n$ on a sheet of parities $\rho, \sigma$ and types $r, s$ is related to the set $n'$ on a $q$-neighbouring sheet of parities $\rho, \sigma'$ and types $r, s'$ by $\sigma' = 1 - \sigma$ and

$$n'_j = n_{s+s'} + n_{s+s'+1} - n_j + \sigma(1 - \delta_{s+s',0}) - F_{j,s+s'}. \tag{14}$$

These recursion relations, together with the conditions

$$m_0 = \cdots m_{N-1} = n_0 \cdots = n_{N-1} = 0 \quad \text{in the initial sheet, where} \quad p, q \in \mathscr{D}_0$$

define the integers $m_0, \dots, m_{N-1}, n_0, \dots, n_{N-1}$ on all sheets. In particular, it follows that

$$\sum_{j=0}^{N-1} m_j = N\mu - r, \qquad \sum_{j=0}^{N-1} n_j = N\nu - s, \tag{15}$$

$\mu$ and $\nu$ being integers.

Define

$$g_k = \sum_{i=0}^{N-1} m_{i+k} n_i. \tag{16}$$

Numerical Fortran experiments (in integer arithmetic) for small $N$ and sheets close to $\mathscr{D}$ strongly suggest that

$$\begin{aligned}
\alpha_j &= \mu - \nu - m_{j+r} - m_{j+r+1} + n_{s-j-1} + n_{s-j} + \tilde{\alpha}_j, \\
\alpha'_j &= -\mu - \nu + m_{j+r} + m_{j+r+1} + \tilde{\alpha}'_j, \\
\beta_j &= \mu + \nu - n_{s-j} - n_{s-j+1} + \tilde{\beta}_j, \\
\beta'_j &= -\mu + \nu + \tilde{\beta}'_j, \\
\gamma_j &= 2\, g_{j+r-s+1} - 2\, g_{j+r-s-1} - 2\, m_{j+r-s+\sigma} + 2\, n_{s-r-j+\rho} \\
&\quad + m_{j+r} + m_{j+r+1} - n_{s-j-1} - n_{s-j} + \tilde{\gamma}_j, \\
\rho_{ij} &= (m_{r-i+1} - m_{r-i-1})\, \delta_{j,0} + (n_{s-j+1} - n_{s-j-1})\, \delta_{i,0} + \tilde{\rho}_{ij},
\end{aligned} \tag{17}$$

where

$$
\begin{aligned}
\tilde{\alpha}_j &= \left[\frac{j}{N}\right]+\left[\frac{j-\rho(1-\sigma)}{N}\right]-\left[\frac{j+r-\rho}{N}\right]-\left[\frac{j-s+\sigma}{N}\right]+(1-\rho)\,\sigma,\\
\tilde{\alpha}'_j &= \left[\frac{j+r-\rho}{N}\right]-\left[\frac{j-s-\rho\sigma}{N}\right]+\rho(1-\sigma),\\
\tilde{\beta}_j &= -\left[\frac{j+r-(1-\rho)(1-\sigma)}{N}\right]+\left[\frac{j-s-1+\sigma}{N}\right]+\rho(1-\sigma),\\
\tilde{\beta}'_j &= \left[\frac{j+r-s-(1-\rho)\,\sigma}{N}\right]-\left[\frac{j}{N}\right]+(1-\rho)\,\sigma,\\
\tilde{\gamma}_j &= \left[\frac{j+r-\rho}{N}\right]+\left[\frac{j-s+\sigma}{N}\right]-\left[\frac{j}{N}\right]-\left[\frac{j+r-s}{N}\right]\\
&\quad -(1-\rho)\,\sigma+\rho(1-\sigma)(-1+2\,\delta_{j,\,s-r}),\\
\tilde{\rho}_{ij} &= \delta_{i,0}\delta_{j,\,s-\sigma}+\delta_{j,0}\delta_{i,\,r-\rho}-\delta_{i,0}\delta_{j,0}.
\end{aligned}
\tag{18}
$$

From (15), incrementing $r(s)$ by $N$ increments $\mu(v)$ by 1, so the right-hand sides of (17) are each periodic in $r$, $s$ and $j$, of period $N$, as they should be.

Define

$$\phi_{\rho\sigma}=\rho+\sigma-2\rho\sigma, \tag{19}$$

so that $\phi_{\rho\sigma}=0$ if $\rho=\sigma$, and $\phi_{\rho\sigma}=1$ if $\rho\neq\sigma$. Our numerical experiments also suggest that,

$$
\begin{aligned}
&\sum_j \alpha_j=\sum_j \beta'_j=(N-1)(1-\rho)\,\sigma+\sum_j\,(n_j-m_j),\\
&\sum_j \alpha'_j=\sum_j \beta_j=(N-1)\,\rho(1-\sigma)+\sum_j\,(m_j-n_j),\\
&\sum_j \gamma_j=-(N-1)\,\phi_{\rho\sigma},
\end{aligned}
\tag{20}
$$

the sums being from $j=0$ to $j=N-1$. It follows that

$$
\begin{aligned}
&\sum_j\,(\alpha_j+\alpha'_j+\gamma_j)=\sum_j\,(\beta_j+\beta'_j+\gamma_j)=0,\\
&\sum_j\,(\alpha_j+\beta_j+\gamma_j)=\sum_j\,(\alpha'_j+\beta'_j+\gamma_j)=0.
\end{aligned}
\tag{21}
$$

We also find that

$$\sum_j j(\alpha_j+\alpha'_j+\beta_j+\beta'_j+\gamma_j)=0, \qquad \text{mod } N \tag{22}$$

if $N$ is odd or if $\rho=\sigma$; if $N$ is even and $\rho\neq\sigma$, then the rhs of (22) equals $N/2$, modulo $N$.

These relations (20)–(22) ensure that no additional external constant factors occur in (9). For instance, multiplying $a_p$ by $\omega^k$, will introduce a factor $\omega^{kL}$, where $L=\sum_j (\alpha_j+\alpha'_j+\gamma_j)$. From (21), this factor is unity. Similarly for $b_p$, $a_q$, $b_q$. Also, interchanging all $p$ variables with the corresponding $q$ variables will introduce a factor $(-1)^I\,\omega^J$, where

$$I=\sum_j (\alpha_j+\alpha'_j+\beta_j+\beta'_j+\gamma_j), \qquad J=\sum_j j(\alpha_j+\alpha'_j+\beta_j+\beta'_j+\gamma_j).$$

From (20) and (22), this factor is also unity.

If we ignore the requirement (15),[3] the relations (18) are unchanged by the substitutions $\rho$, $\sigma$, $r$, $s$, $\mu$, $\nu$, $m_j$, $n_j$, $\alpha_j$, $\alpha'_j$, $\beta_j$, $\beta'_j$, $\gamma_j$, $\rho_{ij}\rightarrow 1-\sigma$, $1-\rho$, $1-s$, $1-r$, $-\nu$, $-\mu$, $-n_{1-j}$, $-m_{1-j}$, $\alpha_j$, $\beta_{j+1}$, $\alpha'_{j-1}$, $\beta'_j$, $\gamma_j$, $\rho_{-j,-i}$.

### *"Dimension" of the Riemann Surface*

We see that the $2N$ integers $m_0,\dots, m_{N-1}$, $n_0,\dots, n_{N-1}$ are sufficient to specify the function $T_{pq}$ on any sheet within its Riemann surface, so any sheet can be associated with a point in a $2N$-dimensional space. Further, incrementing each of $m_0,\dots, n_{N-1}$ (and therefore also $\mu$, $\nu$) by unity (or any integer) leaves (17) unchanged, so the space can be reduced to one of dimension $2N-1$. This appears to be a partial analogue for the chiral Potts model of the rapidity "difference property" that plays such an important role in the simpler models.

For $N>2$ this appears to be the best one can do—each sheet of the Riemann surface corresponds to a point in a $(2N-1)$-dimensional space. Note however that sheets are neighbours if their $m$-integers satisfy (12), or if their $n$-integers satisfy (12). Hence neighbouring sheets do not necessarily correspond to neighbouring points in $m_0,\dots, n_{N-1}$-space.

For $N=2$ we have the additional relations $m_{1-\rho}=n_{1-\sigma}=0$ for all sheets, so $m_0$, $m_1$, $n_0$, $n_1$ enter (17) only via $m_\rho-n_\sigma$ and the space is one-dimensional, as we observed in ref. 6.

[3] Presumably there is some variant of this symmetry that preserves the equations (17) completely.

For $N>2$ we note that the exponents $\alpha_j,\ldots,\beta'_j,\rho_{ij}$ are linear in the integers $m_0,\ldots,n_{N-1}$, but the $\gamma_j$ are linear only separately in $m_0,\ldots,m_{N-1}$ and $n_0,\ldots,n_{N-1}$. They are bilinear in the full set of $2N$ integers, due to the occurrence of the $g_j$, as defined by (16).

The values of $p=\{x_p, y_p, \rho, r, m_0,\ldots, m_{N-1}\}$ completely specify the point $p$, not only on the algebraic curve (2), but also on the Riemann surface of $T_{pq}$. Similarly, $q=\{x_q, y_q, \sigma, s, n_0,\ldots, n_{N-1}\}$ completely specifies $q$. We shall refer to the corresponding Riemann sheet as the "sheet $(m, n)$."

## 4. ZEROS AND POLES OF $T_{pq}$

For $a_p$, $b_p$, $a_q$, $b_q$ in the central domain $\mathscr{D}$, the functions $T(\omega^i a_p, b_p \,|\, \omega^j a_q, b_q)$ on the rhs of (9) are non-zero and analytic. Hence the rhs of (9) is meromorphic in $\mathscr{D}$, with zeros or poles only when

$$a_p=\omega^j a_q \qquad \text{and} \qquad b_p=b_q, \tag{23}$$

which implies $a_p b_p=\omega^j a_q b_q$. Thus its zero at this point is of order

$$\alpha_j+\beta'_0+\gamma_j. \tag{24}$$

(Equivalently, its pole is of order $-\alpha_j-\beta'_0-\gamma_j$.)

From (2), the relation $x_p^N=x_q^N$ implies $y_p^N=y_q^N$ and vice-versa, and either implies $t_p^N=t_q^N$. Similarly, $x_p^N=y_q^N$ implies $y_p^N=x_q^N$ and again $t_p^N=t_q^N$. From (8) it follows that the function $T(x_p, y_p \,|\, x_q, y_q)$ is meromorphic throughout its Riemann surface, with zeros and poles only when $t_p^N=t_q^N$. On the sheet $(\rho, \sigma, r, s, m, n)$ the zero at $t_p=\omega^k t_q$ $(k=0,\ldots, N-1)$ is of order

$$z_k=(-1)^{\rho+\sigma}\,(\alpha_{k+s-r}+\beta'_0+\gamma_{k+s-r}).$$

From (17) and (18) it follows that

$$(-1)^{\rho+\sigma}\, z_k=2g_{k+1}-2g_{k-1}-2m_{k+\sigma}+2n_{\rho-k}+2\rho(1-\sigma)(\delta_{k,0}-1)+w_k, \tag{25}$$

where

$$w_k=\left[\frac{k+s-r-\rho(1-\sigma)}{N}\right]-\left[\frac{k}{N}\right]+\left[\frac{r-s-(1-\rho)\,\sigma}{N}\right]+\phi_{\rho\sigma}. \tag{26}$$

The contribution $w_k$ has a simple explanation. Using the notation of refs. 2 and 8, let us postulate the existence of functions $\Theta_{i,j}$, $\bar{\Theta}_{i,j}$ of $p$ and $q$ such that $\Theta_{i,j}$ has simple zeros on the Riemann surface when

$$x_q = \omega^i y_p \qquad \text{and} \qquad y_q = \omega^j x_p, \tag{27}$$

and $\bar{\Theta}_{i,j}$ has simple zeros when

$$x_q = \omega^i x_p \qquad \text{and} \qquad y_q = \omega^j y_p. \tag{28}$$

Any zero or pole of $T_{pq}$ must occur at one of these points on some particular Riemann sheet.

Define $G_{pq}$ by

$$G_{pq} = G(x_q, y_q) = \prod_{i=1}^{N-1} \prod_{j=1}^{N-i} [\Theta_{N-i,N+1-j}\, \bar{\Theta}_{i,j}]^{-1}, \tag{29}$$

Then $G_{pq}$ is the function $1/\xi\bar{\xi}$ of ref. 2. We remark therein that its poles contain just all the poles of the Boltzmann weight functions $W_{pq}(n)$, $\bar{W}_{pq}(n)$. Hence for a finite square lattice of $\mathcal{N}$ sites with partition function $Z$, it is true that $Z/G_{pq}^{\mathcal{N}}$ has no poles on the Riemann surface.

It is therefore natural to define a normalized function $\tilde{T}_{pq}$ by

$$T_{pq} = G_{pq} \tilde{T}_{pq}. \tag{30}$$

$\tilde{T}_{pq}$ is then the partition function per site in the normalization where the Boltzmann weights are free of poles, and where there are no zeros common to $W_{pq}(0),\ldots, W_{pq}(N-1)$, and none common to $\bar{W}_{pq}(0),\ldots, \bar{W}_{pq}(N-1)$.

On the sheet $(\rho, \sigma, r, s, m, n)$ we find from (29) that the pole of $G_{pq}$ at $t_p = \omega^k t_q$ $(k = 0,\ldots, N-1)$ is of order $-(-1)^{\rho+\sigma} w_k$ (so numerically this is either 0 or 1). Hence the term $w_k$ in (25) is the contribution of $G_{pq}$. The preceding terms (all even integers) are the contribution of $\tilde{T}_{pq}$.

Let $\theta_k$ be the function with simple zeros on the Riemann surface when $t_p = \omega^k t_q$ and $x_p^N = y_q^N$, $y_p^N = x_q^N$. Then

$$\theta_k \sim \prod_{i=0}^{N-1} \Theta_{i,-k-i}, \tag{31}$$

where by $f \sim g$ we mean herein that $f$ and $g$ have the same zeros and poles (when $t_p^N = t_q^N$) on the Riemann surface.

Similarly,

$$\bar{\theta}_k \sim \prod_{i=0}^{N-1} \bar{\Theta}_{i,-k-i}, \tag{32}$$

where $\bar{\theta}_k$ has simple zeros when $t_p = \omega^k t_q$ and $x_p^N = x_q^N$, $y_p^N = y_q^N$. Then formally we can write (25) as

$$\tilde{T}_{pq} \sim \prod (\bar{\theta}_k / \theta_k)^{2\epsilon_{pq}(k)}, \tag{33}$$

where the product is over all Riemann sheets $(\rho, \sigma, r, s, m, n)$ and all values $0, \ldots, N-1$ of $k$, and the integer $\epsilon_{pq}(k)$ is

$$\epsilon_{pq}(k) = g_{k+1} - g_{k-1} - m_{k+\sigma} + n_{\rho-k} + \rho(1-\sigma)(\delta_{k,0} - 1). \tag{34}$$

If $\rho = \sigma$ then only $\bar{\theta}_k$ can have zeros, and if $\rho \neq \sigma$ only $\theta_k$.

We note that there are various formal identities between our $\Theta$ and $\theta$ functions, notably

$$\prod_j \bar{\Theta}_{ij} \sim x_q - \omega^i y_p, \quad \prod_i \bar{\Theta}_{ij} \sim y_q - \omega^j x_p, \quad \prod_j \Theta_{ij} \sim x_q - \omega^i x_p$$
$$\prod_i \Theta_{ij} \sim y_q - \omega^j y_p, \qquad \theta_k \bar{\theta}_k \sim t_p - \omega^k t_q, \tag{35}$$

all products being over the integers $0, \ldots, N-1$.

It may be possible to give explicit representations of our postulated functions $\Theta_{ij}$, $\bar{\Theta}_{ij}$, $G_{pq}$, $\theta_k$, $\bar{\theta}_k$, $\tilde{T}_{pq}$ in terms of hyperelliptic functions,[9–11] but we shall not do so here. Using them does greatly simplify the relations and certainly provides a way of keeping track of the poles and zeros at $t_q^N = t_p^N$ on the Riemann surface. These are the only poles and zeros, apart possibly from ones occurring when $p$ has some particular value independent of $q$, or $q$ has some value independent of $p$. This suggests that herein the relation

$$f_{pq} \sim g_{pq}$$

is equivalent to the explicit identity

$$\frac{f_{pq} f_{p'q'}}{f_{pq'} f_{p'q}} = \frac{g_{pq} g_{p'q'}}{g_{pq'} g_{p'q}},$$

for all rapidities $p, q, p', q'$. This implies that there exist single-rapidity functions $u_p$, $v_q$ such that

$$f_{pq} = u_p g_{pq} v_q.$$

In all the cases where we have been able to test this hypothesis, e.g., by using (35), we have found it to be true.

## 5. AUTOMORPHISMS

Three basic automorphisms that take a point on the curve (2) to another such point are $R$, $S$, $U$, where

$$R\colon x_{Rp} = y_p, \quad y_{Rp} = \omega x_p; \qquad S\colon x_{Sp} = y_p^{-1}, \quad y_{Sp} = x_p^{-1}; \qquad U\colon x_{Up} = \omega x_p, \quad y_{Up} = y_p. \tag{36}$$

If $p$ lies in the initial domain $\mathscr{D}_0$ (so $y_p \simeq 1$), then we can take $Up$ to also lie in $\mathscr{D}_0$; and $Rp$, $Sp$ to lie in the adjacent domain $\mathscr{D}'_0$.[4]

We can determine what happens when $p$ is not in the initial domain $\mathscr{D}_0$ by analytic continuation. If $p$ lies on a sheet that is a $k$th neighbour of $\mathscr{D}_0$, a route to it being the sequence (4) of sheets of types $\{0, r_1, r_2,..., r_k\}$, then $Up$ is also on a $k$th neighbouring sheet (so has the same parity $\rho$), but from (36) the sequence to $Up$ is $\{0, r_1+1, r_2, r_3+1,..., r_k+\rho\}$.[5]

As above, writing $p = \{x_p, y_p, \rho, r, m_j\}$ for the full set of parameters defining the rapidity $p$ (with $j = 0,..., N-1$), it follows that

$$Up = \{\omega x_p, y_p, \rho, r+\rho, m_{j-1} + (-1)^\rho (m_{N-1} - m_0) - \rho\delta_{j,1}\}. \tag{37}$$

Iterating, we obtain for all integers $i$

$$U^i p = \{\omega^i x_p, y_p, \rho, r+i\rho, m_{j-i} + (-1)^\rho (m_{N-i} - m_0) - \rho F_{j,i}\}. \tag{38}$$

The $r$ values should always be taken modulo $N$, so that $0 \leqslant r < N$. Then we see, using (13), that $U^N p = p$.

Similarly, $Rp$ and $Sp$ lie on $(k+1)$-th neighbouring sheets of $\mathscr{D}_0$, at the termini of the routes $\{0, 0, r_1+1, r_2, r_3+1,..., r_k+\rho\}$, $\{0, 0, -r_1, -r_2, -r_3,..., -r_k\}$, respectively, and

$$Rp = \{y_p, \omega x_p, 1-\rho, r+\rho, m_{j-1} + \rho(1-\delta_{j,1})\}, \tag{39}$$

$$Sp = \{1/y_p, 1/x_p, 1-\rho, -r, -m_{N+1-j}\}. \tag{40}$$

[4] $R$ and $S$ are the same as in ref. 12, while $R$, $U$ occur in ref. 13.

[5] Note that in general $Up$ is *not* obtained from $p$ by staying on the same sheet and simply replacing $x_p$ by $\omega x_p$.

Two combinations of automorphisms that we shall need are $V = RU^{-1}$ and $U^i V$:

$$Vp = RU^{-1}p = \{y_p, x_p, 1-\rho, r, m_j + (-1)^\rho (m_1 - m_0)\} \tag{41}$$

and

$$U^i V p = \{\omega^i y_p, x_p, 1-\rho, r + i(1-\rho), m_{j-i} + (-1)^\rho (m_1 - m_{N-i}) - (1-\rho) F_{j,i}\} \tag{42}$$

for any integer $i$.

The effect of these automorphisms on the second rapidity $q$ can be obtained at once by replacing $p, \rho, r, m_0,..., m_{N-1}$ by $q, \sigma, s, n_0,..., n_{N-1}$.

## 6. RELATIONS FOR $\tilde{T}_{pq}$ AND ITS EXPONENTS

Going to the pole-free normalization (30), the relations (5)–(7) simplify to

$$\tilde{T}_r(\omega^r y_p, x_p \mid x_q, y_q) \sim \frac{\tilde{T}(\omega^r x_p, y_p \mid x_q, y_q)\, \bar{\theta}_1^2 \bar{\theta}_2^2 \cdots \bar{\theta}_{N-r-1}^2 \theta_{N-r}^2 \cdots \theta_{N-1}^2}{\tilde{T}(\omega^{-1} x_p, y_p \mid x_q, y_q)\, \tilde{T}(x_p, y_p \mid x_q, y_q)} \tag{43}$$

$$\tilde{T}_s(x_p, y_p \mid \omega^s y_q, x_q) \sim \frac{\tilde{T}(x_p, y_p \mid \omega^s x_q, y_q)\, \bar{\theta}_0^2 \bar{\theta}_1^2 \cdots \bar{\theta}_{s-1}^2 \theta_s^2 \theta_{s+1}^2 \cdots \theta_{N-2}^2}{\tilde{T}(x_p, y_p \mid \omega^{-1} x_q, y_q)\, \tilde{T}(x_p, y_p \mid x_q, y_q)} \tag{44}$$

$$\frac{\tilde{T}(x_p, y_p \mid x_q, y_q)\, \tilde{T}(\omega x_p, y_p \mid \omega x_q, y_q)}{\tilde{T}(x_p, y_p \mid \omega x_q, y_q)\, \tilde{T}(\omega x_p, y_p \mid x_q, y_q)} \sim (\theta_{N-1}/\theta_0)^2. \tag{45}$$

Again, in these relations as written, $p, q$ both lie in the central domain $\mathscr{D}_0$. However, we can now analytically continue to any Riemann sheet and use the above automorphisms to obtain

$$\begin{aligned} \tilde{T}_{U^r V p, q} &\sim \tilde{T}_{U^r p, q} \bar{\theta}_1^2 \bar{\theta}_2^2 \cdots \bar{\theta}_{N-r-1}^2 \theta_{N-r}^2 \theta_{N-r+1}^2 \cdots \theta_{N-1}^2 / (\tilde{T}_{U^{-1} p, q}\, \tilde{T}_{pq}), \\ \tilde{T}_{p, U^s V q} &\sim \tilde{T}_{p, U^s q} \bar{\theta}_0^2 \bar{\theta}_1^2 \cdots \bar{\theta}_{s-1}^2 \theta_s^2 \theta_{s+1}^2 \cdots \theta_{N-2}^2 / (\tilde{T}_{p, U^{-1} q} \tilde{T}_{pq}), \\ \tilde{T}_{pq}\, \tilde{T}_{Up, Uq} / (\tilde{T}_{p, Uq}\, \tilde{T}_{Up, q}) &\sim (\theta_{N-1}/\theta_0)^2 \end{aligned} \tag{46}$$

for all $p, q$. Note that in these relations $r, s$ must both have values in the set $\{0, 1,..., N-1\}$.

Substituting the form (33) of $\tilde{T}_{pq}$, we obtain the exponent relations

$$\begin{aligned}
-\epsilon_{U^rVp,\,q}(k+r) &= \epsilon_{U^rp,\,q}(k+r)-\epsilon_{U^{-1}p,\,q}(k-1)-\epsilon_{pq}(k)\\
&\quad+\left[\frac{k+(N-1)(1-\phi_{\rho\sigma})}{N}\right]-\left[\frac{k+r}{N}\right]+\left[\frac{r}{N}\right],\\
-\epsilon_{p,\,U^sVq}(k-s) &= \epsilon_{p,\,U^sq}(k-s)-\epsilon_{p,\,U^{-1}q}(k+1)-\epsilon_{pq}(k)\\
&\quad+\left[\frac{k+(N-1)\,\phi_{\rho\sigma}}{N}\right]-\left[\frac{k-s}{N}\right]-\left[\frac{s}{N}\right],\\
\epsilon_{pq}(k)+\epsilon_{Up,\,Uq}(k) &- \epsilon_{Up,\,q}(k+1)-\epsilon_{p,\,Uq}(k-1)=\phi_{\rho\sigma}(\delta_{k,\,0}-\delta_{k,\,N-1}).
\end{aligned} \tag{47}$$

We have used Mathematica to verify for $N=2,\dots,12$ that these equations are indeed satisfied by (34). They are explicitly periodic in $r$, $s$, of period $N$, so are true for all integers $r$, $s$.

For future reference, we note that the half-exponents for $\tilde{T}_{p,\,U^\alpha q}$ are

$$\epsilon_{p,\,U^\alpha q}(k-\alpha)=\psi(\rho,\sigma,k,\alpha\,|\,m,n), \tag{48}$$

where

$$\begin{aligned}
\psi(\rho,\sigma,k,\alpha\,|\,m,n) &= g_{k+1}-g_{k-1}+n_{\rho-k}-\sigma(m_k+m_{k+1})\\
&\quad+(-1)^\sigma\,(n_{N-\alpha}-n_0-m_{k-\alpha})-\sigma F_{\rho-k+\alpha,\,\alpha}+\rho(1-\sigma)(\delta_{k,\,\alpha}-1).
\end{aligned} \tag{49}$$

Also, those for and $\tilde{T}_{U^\alpha p,\,q}$ are

$$\begin{aligned}
\epsilon_{U^\alpha p,\,q}(k+\alpha) &= g_{k+1}-g_{k-1}-m_{k+\sigma}+\rho(n_{-k}+n_{1-k})\\
&\quad+(-1)^\rho\,(m_0-m_{N-\alpha}+n_{-k-\alpha})+\rho F_{k+\alpha+\sigma,\,\alpha}+\rho(1-\sigma)(\delta_{k,\,-\alpha}-1).
\end{aligned} \tag{50}$$

## 7. THE FUNCTION $\tau_2(p, q)$

In the derivation of $T_{pq}$ given in refs. 2 and 3, an important role is played by the auxiliary function $\tau_2(p,q)$. Noting that the $T(x_q, y_q)$ in ref. 6 is $T_{pq}$, while $T(\omega x_q, y_q)$ is $T_{p,\,Uq}$, it follows that Eq. (25) of ref. 6 can be written, for all Riemann sheets, as

$$\tau_2(p,q)=\left[\frac{(y_p-\omega x_q)(t_p-t_q)}{y_p^2\,(x_p-x_q)}\right]^L\frac{T_{pq}}{T_{p,\,Uq}} \tag{51}$$

writing $\tau_2(t_q)$ in ref. 6 as $\tau_2(p, q)$. Also, in (53) of ref. 6 we obtain the result

$$\log \tau_2(p, q) = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{1+\lambda_p e^{i\theta}}{1-\lambda_p e^{i\theta}} \right) \log \left[ \frac{\Delta(\theta) - \omega t_q}{y_p^2} \right] d\theta \tag{52}$$

for $p$, $q$ both in the central domain $\mathscr{D}_0$, where $|\lambda_p|$, $|\lambda_q| < 1$. Here

$$\Delta(\theta) = \left( \frac{1 - 2k' \cos\theta + k'^2}{k^2} \right)^{1/N} \tag{53}$$

Setting $j = N$ in (14) of ref. 6 and using (21) and (23) therein, we find

$$T_{pq} T_{p,Vq} = \frac{N\,(y_p - x_q)(y_p - y_q)(t_p^N - t_q^N)}{y_p^2\,(x_p^N - x_q^N)(y_p^N - y_q^N)\,\tau_2(p, U^{-1}q)} \tag{54}$$

for all Riemann sheets. The LHS is the free energy (per double site) of a model with vertical rapidities $p$ and alternating horizontal rapidities $q$, $Vq$. This is the general "superintegrable" model discussed in ref. 14, so we see that to within simple known scalar factors, $1/\tau_2(p, U^{-1}q)$ is the free energy of this model. Indeed $\tau_2(p, U^{-1}q)$ itself is the free energy (apart possibly from simple scalar factors) of the "inverse" model introduced in ref. 14, while $\tau_2(p, q)$ is the free energy of the model defined in (3.44)–(3.48) of ref. 13 (with $j = 2$ and $k = 0$).

We can take (51) as the definition of $\tau_2(p, q)$. It follows at once that

$$\tau_2(p, q)\,\tau_2(p, Uq) \cdots \tau_2(p, U^{N-1}q) = \frac{(y_p^N - x_q^N)(t_p^N - t_q^N)}{y_p^{2N}(x_p^N - x_q^N)} \tag{55}$$

Also, considered as a function of $t_q$, $\tau_2(p, q)$ only has a single branch cut, from $\omega^{-1}\eta$ to $\omega^{-1}/\eta$. Across the other $N-1$ potential branch cuts it is in fact an analytic function of $t_q$. Together with (55), this implies that

$$\tau_2(p, U^i V U^{-i} q) = v_{pq}^{\delta_{i,N-1}}\, \tau_2(p, q), \tag{56}$$

where

$$v_{pq} = \frac{(x_p^N - x_q^N)\,(y_p^N - y_q^N)}{(x_p^N - y_q^N)\,(y_p^N - x_q^N)}. \tag{57}$$

(Going from $q$ to $U^i V U^{-i} q$ takes one from $\mathscr{D}_0$ to the neighbouring domain $\mathscr{D}'_i$, while leaving $t_q$ unchanged.)

One can also verify directly from (55) that

$$\tau_2(p, q)\,\tau_2(U^i V U^{-i} p, q) = (\omega^{-i} t_p - \omega t_q)^2 / y_p^4. \tag{58}$$

*Exponent Relations*

The exponents of $\tau_2(p, q)$ are simpler than those of the free energy function $T_{pq}$, from (51) and (33),

$$\tau_2(p, q) \sim \theta_0^2\, \tilde{T}_{pq}/\tilde{T}_{p,\,Uq} \sim \prod\, (\bar{\theta}_k/\theta_k)^{2\tilde{\epsilon}_{pq}(k)}, \tag{59}$$

where

$$\begin{aligned}\tilde{\epsilon}_{pq}(k) &= \epsilon_{pq}(k) - \epsilon_{p,\,Uq}(k-1) - \phi_{\rho\sigma}\delta_{k,\,0}\\ &= (-1)^\sigma\,(m_{k-1} - m_k + n_0 - n_{N-1} - \rho\delta_{k,\,1}).\end{aligned} \tag{60}$$

We see that the bilinear terms $g_k$ do not occur in $\tilde{\epsilon}_{pq}(k)$, leaving only terms that are fully linear in the $m_j$ and $n_j$.

The Eqs. (55), (56), and (58) imply that the exponents $\tilde{\epsilon}_{pq}(k)$ satisfy

$$\begin{aligned}\tilde{\epsilon}_{pq}(k) + \tilde{\epsilon}_{p,\,Uq}(k-1) + \cdots + \tilde{\epsilon}_{p,\,U^{N-1}q}(k-N+1) &= -\phi_{\rho\sigma},\\ -\tilde{\epsilon}_{p,\,U^iVU^{-i}q}(k) &= \tilde{\epsilon}_{pq}(k) + \delta_{i,\,N-1},\\ \tilde{\epsilon}_{pq}(k) - \tilde{\epsilon}_{U^iVU^{-i}p,\,q}(k) &= (1-2\,\phi_{\rho\sigma})\,\delta_{k,\,i+1}\end{aligned} \tag{61}$$

for all $\rho, \sigma, k, m_0,..., m_{N-1}, n_0,..., n_{N-1}$. Indeed we find, using (60) and (37), that this is so.

The half-exponents for $\tilde{\epsilon}_{p,\,U^\alpha q}$ are

$$\tilde{\epsilon}_{p,\,U^\alpha q}(k-\alpha) = (-1)^\sigma\,(m_{k-\alpha-1} - m_{k-\alpha} + n_{-\alpha} - n_{-\alpha-1} - \rho\delta_{k,\,\alpha+1} + \sigma\delta_{\alpha,\,N-1}). \tag{62}$$

## 8. "SUFFICIENCY" OF THE ROTATION AND INVERSION RELATIONS

As is consistent with series expansions, the function $T_{pq}$ is non-zero and analytic in the central (physical) domain $\mathscr{D}_0$. Given this, the recursion relations (5)–(7) are certainly sufficient to determine the Riemann surface on which $T_{pq}$ lives, that $T_{pq}$ is meromorphic on this surface, and to give the orders (exponents) $\epsilon_{pq}(k)$ of all its zeros and poles on every sheet.

This goes a long way towards defining $T_{pq}$. To complete the description one needs to establish that the ratio of two such functions with the same zeros and poles has some periodicity property from sheet to sheet so that it is bounded over the whole surface. It is certainly entire, so by Liouville's theorem it would then have to be a constant. Such constants can usually be fixed from special cases.

We shall not discuss this problem of completing the description further herein, but will suppose that it can be done. Here our concern is to see if the weaker rotation and inversion relations can be used to determine the exponents $\epsilon_{pq}(k)$.

More specifically, the rotation and inversion relations, together with the analyticity properties in the central domain, are known to be sufficient to determine the free energy for the two-dimensional lattice models with the "rapidity difference property."[6]

The chiral Potts model does not possess the rapidity difference property, and there is no parametrization in terms of single-argument Jacobi elliptic functions. Nevertheless, we can ask whether its rotation and inversion relations, together with some simple and plausible ansatz, are sufficient to determine the free energy exponents $\epsilon_{pq}(k)$. This is the question we address in the remainder of this paper.

### *The Relations*

From Eqs. (39), (40), and (10) of ref. 6, the rotation and inversion relations are

$$T_{q,Rp} = T_{pq}, \qquad T_{qp}T_{pq} = r_{pq}, \tag{63}$$

where

$$r_{pq} = \frac{N\,(x_p - x_q)(y_p - y_q)(t_p^N - t_q^N)}{(x_p^N - x_q^N)(y_p^N - y_q^N)(t_p - t_q)}. \tag{64}$$

From these we can deduce that

$$T_{Rp,q}T_{pq} = r_{Rp,q}, \qquad T_{p,Rq}T_{pq} = r_{pq}. \tag{65}$$

These two equations can be obtained from (5) and (6) by setting $r = s = 0$ and replacing either $x_p$ or $x_q$ by $\omega x_p$ or $\omega x_q$. They can also be obtained by setting $r = s = N-1$ and replacing either $p$ or $q$ by $Rp$ or $Rq$. There is therefore an overlap between (63) and the recursion relations (5) and (6), but the latter can not be deduced from the former.

Using (29), (30), and (63) become

$$\tilde{T}_{q,Rp} \sim \tilde{T}_{pq}, \qquad \tilde{T}_{qp}\tilde{T}_{pq} \sim (\theta_1\theta_2\cdots\theta_{N-1})^2. \tag{66}$$

[6] This property implies that such a model can be parametrized in terms of single-argument Jacobi elliptic functions, which fixes the Riemann surface for the free energy.

On a given Riemann sheet of $p$-parity and type $\rho, r$, and $q$-parity and type $\sigma, s$, specified by the integers $m_0, \ldots, m_{N-1}$, $n_0, \ldots, n_{N-1}$, let the order of the zero when $t_p = \omega^k t_q$ be $2(-1)^{\rho+\sigma} e(\rho, \sigma, k, r, s \,|\, m, n)$, where $m = \{m_0, \ldots, m_{N-1}\}$ and $n = \{n_0, \ldots, n_{N-1}\}$. Then, analogously to (33), we can write

$$\tilde{T}_{pq} \sim \prod (\bar{\theta}_k / \theta_k)^{2e(\rho, \sigma, k, r, s \,|\, m, n)}, \tag{67}$$

the product being over all zeros (and poles), and all Riemann sheets.

Are the rotation and inversion relations (66) sufficient to fix $e(\rho, \sigma, k, r, s \,|\, m, n)$ as the $\epsilon_{pq}(k)$ given by (34)? First we must note that there are severe self-consistency restrictions on the exponents $e(\rho, \sigma, k, r, s \,|\, m, n)$, for any meromorphic function on the Riemann surface.

## *Consistency*

Consider some particular zero, at $t_p = \omega^k t_q$, on some particular sheet of types $r, s$. For definiteness, take the parities $\rho, \sigma$ to be zero. Then (since $y_p \simeq \omega^r$, $y_q \simeq \omega^s$) the zero is at $x_p = \omega^{k+s-r} x_q$, $y_p = \omega^{r-s} y_q$.

Now move $p$ and $q$ to adjacent sheets of types $r', s'$, respectively, so their parities both become one. Now consider the zero at $t_p = \omega^{k'} t_q$. This must be when $x_p = \omega^{r'-s'} x_q$ and $y_p = \omega^{k'+s'-r'} y_q$. If $k = k' = r + r' - s - s'$, this is the same zero as the one on the previous sheet: we have simply followed it from one sheet to the next. Its exponent (an integer) must be the same, so

$$e(\rho, \sigma, k, r, s \,|\, m, n) = e(1-\rho, 1-\sigma, k, r', s' \,|\, m', n'), \tag{68}$$

provided $k = r + r' - s - s'$ and $m', n'$ are given by (12) and (14). We have only established this condition for $\rho = \sigma = 0$, but the same result is obtained for all $\rho, \sigma$.

This is a very strong condition on the exponents $e(\rho, \sigma, k, r, s \,|\, m, n)$. It must be true for all values of $\rho, \sigma, r, s, r', s'$, and all values of the integers $m_j, n_j$. Furthermore, it must be true for any meromorphic function on the Riemann surface. Hence it is true for the exponents $\tilde{\epsilon}_{pq}(k)$ of $\tau_2(p, q)$ as well as those of $T_{pq}$ and $\tilde{T}_{pq}$.

## *Rotation and Inversion*

From (27)–(32), exhibiting the dependence of $\theta_k, \bar{\theta}_k$ on $p, q$:

$$[\theta_k]_{qp} = [\theta_{-k}]_{pq}, \qquad [\bar{\theta}_k]_{qp} = [\bar{\theta}_{-k}]_{pq}, \tag{69}$$

$$[\theta_k]_{q, Rp} = [\bar{\theta}_{-k-1}]_{pq}, \qquad [\bar{\theta}_k]_{q, Rp} = [\theta_{-k-1}]_{pq}. \tag{70}$$

Together with (67), (66), and (39), these imply the relations

$$e(\rho, \sigma, k, r, s \mid m, n) = -e(\sigma, 1-\rho, -k-1, s, r+\rho \mid n, Rm), \tag{71}$$

$$e(\rho, \sigma, k, r, s \mid m, n) + e(\sigma, \rho, -k, s, r \mid n, m) = \phi_{\rho\sigma}(\delta_{k,0} - 1), \tag{72}$$

true for all $\rho, \sigma, k, r, s, m, n$. Here $Rm$ is the result of the operation $R$ on the set $m$, so from (39) it follows that $(Rm)_j = m_{j-1} + \rho(1-\delta_{j,1})$.

### Other Elementary Relations

There are no zeros or poles of $T_{pq}$ or $\tilde{T}_{pq}$ in the central domain $\mathscr{D}_0$, which is when $m = n = 0$, so

$$e(0, 0, k, 0, 0, \mid 0, 0) = 0. \tag{73}$$

The model also possesses a reflection symmetry.[12] Let $S$ be the automorphism defined in (36) and (40). Then the Boltzmann weights satisfy $W_{Sq,Sp}(n) = W_{pq}(n)$, $\bar{W}_{Sq,Sp}(n) = \bar{W}_{pq}(-n)$. Replacing $p, q$ by $Sq, Sp$ therefore reflects the lattice about its SW–NE axis. This does not change the partition function, so $T_{Sq,Sp} = T_{pq}$.

It also leaves $\Theta_{ij}$ unchanged, while replacing $\bar{\Theta}_{ij}$ by $\bar{\Theta}_{ji}$. The product over $i, j$ in (29) is symmetric in $i, j$, so $G_{pq}$ is unchanged and from (30)

$$\tilde{T}_{Sq,Sp} = \tilde{T}_{pq}. \tag{74}$$

From (67) it follows that

$$e(1-\sigma, 1-\rho, k, -s, -r \mid Sn, Sm) = e(\rho, \sigma, k, r, s \mid m, n), \tag{75}$$

where $Sm$ is $S$ acting on the set $m$, so from (40) $(Sm)_j = -m_{N+1-j}$. Similarly for $Sn$.

### A Bilinear Ansatz

To give the free energy, the rotation and inversion relations must always be supplemented by analyticity assumptions, We know from the result (34) that $e(\rho, \sigma, k, r, s \mid m, n)$ is linear in the $m_j$ and $n_j$ separately, and hence bilinear in their combination. This seems to be a basic property that one may have expected, so we assume that there exist coefficients $A, B, C, D$ such that

$$\begin{aligned} e(\rho, \sigma, k, r, s \mid m, n) &= A(\rho, \sigma, k, r, s) + \sum_{j=0}^{N-1} [B(\rho, \sigma, k, r, s \mid j)\, m_j + C(\rho, \sigma, k, r, s \mid j)\, n_j] \\ &\quad + \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} D(\rho, \sigma, k, r, s \mid i, j)\, m_i n_j \end{aligned} \tag{76}$$

for all integers $m_j, n_j$. For generality we allow the coefficients to explicitly depend on the types $r, s$ of the Riemann sheet.

There are thus $4N^3(N+1)^2$ unknown coefficients to determine. Rather than proceed fully algebraically, we have looked at modest values of $N$ (not bigger than 12) and used Mathematica (with $m_j, n_j$ arbitrary) to determine the number of variables fixed by the various equations. Thus the following remarks are extrapolated conjectures from our observations.

We take $N \geqslant 3$. As we noted above, the Ising case $N=2$ is special in that then the $m_j$ and $n_j$ are not linearly independent, but satisfy $m_{1-\rho} = n_{1-\sigma} = 0$. All the solutions we write down are valid for $N=2$, but there may be other solutions, so that our remarks concerning the number of solutions may not apply.

First we substituted the ansatz (76) into the consistency condition (68) and observed that the number of undetermined coefficients reduced to $2N^2(N+1)$. Since one can fix $k, r-s$ and $(-1)^{\rho+\sigma}$ in this condition, this means that for each such case there are just $N+1$ unknown coefficients, i.e., $N+1$ linearly independent functions satisfying (68).

We know what these functions are. The functions $\tilde{T}_{p, U^\alpha q}$, $\tilde{T}_{U^\alpha p, q}$, $\tau_2(p, U^\alpha q)$ are all meromorphic on the Riemann surface. Their exponents, given by (49), (50), and (62), must therefore satisfy (68). Taking $\alpha = 0, \ldots, N-1$, we obtain $3N$ solutions of (68). They cannot all be linearly independent, but those in the first set certainly are, so that gives us $N$ solutions. The remaining one is a constant—independent of the $m_j$ and $n_j$.

It follows that $e(\rho, \sigma, k, r, s \,|\, m, n)$ must be of the form

$$e(\rho, \sigma, k, r, s \,|\, m, n) = \gamma(\rho, \sigma, k, r-s) + \sum_{\alpha=0}^{N-1} c(\rho, \sigma, k, r-s \,|\, \alpha)\, \psi(\rho, \sigma, k, \alpha \,|\, m, n) \qquad (77)$$

where the coefficients $\gamma$, $c$ satisfy

$$\gamma(\rho, \sigma, k, \lambda) = \gamma(1-\rho, 1-\sigma, k, k-\lambda),$$
$$c(\rho, \sigma, k, \lambda, \alpha) = c(1-\rho, 1-\sigma, k, k-\lambda, \alpha),$$

so there are $2N^2(N+1)$ independent coefficients $\gamma$, $c$, as yet undetermined. The consistency condition (68) is now satisfied.

For $N$ odd, the rotation relation (71) reduces the number of undetermined coefficients to $N(N+1)$. Then the inversion relation (72) fixes all the coefficients, giving the solution (34). The extra relations (73), (75) are then satisfied.

For $N$ even and greater than 2, (71) reduces the number of undetermined coefficients to $N(5N+6)/4$. Then (72) further reduces the number to $[(N+4)/4]$ (so for $N=$ 4, 6, 8, 10, 12 the numbers are 2, 2, 3, 3, 4). The other relations (73) and (75) are satisfied by this solution, so do not reduce the number of undetermined coefficients any further. The solution is then of the form

$$\epsilon_{pq}(k)+h(\rho-\sigma-k+2r-2s)\, f(\rho,\sigma,k\,|\,m,n),$$

where $\epsilon_{pq}(k)$ is given by (34) and

$$\begin{aligned} f(\rho,\sigma,k\,|\,m,n) = {} & g_{k+1}-g_{k-1}-m_{k+\sigma}+n_{\rho-k} \\ & +\sum_{\alpha=0}^{N-1}\left[(-1)^{\alpha-k-\sigma}\, m_\alpha-(-1)^{\alpha+k-\rho}\, n_\alpha\right]. \end{aligned}$$

These functions have the weak difference property that they are unchanged (for $N$ even) by incrementing all of $m_0,\ldots,m_{N-1},n_0,\ldots,n_{N-1}$ by the same arbitrary integer, so no further restrictions can be obtained by making this requirement.

The coefficients $h(j)$ are integers, subject only to the constraints

$$h(j)=h(N-j)=h(N+j), \qquad \text{and} \qquad h(j)=0 \qquad \text{if} \quad j \quad \text{is odd.}$$

It follows that just $[(N+4)/4]$ of them (all with $j$ even) remain undetermined.

To summarize: if $N$ is odd, the rotation and inversion relations, together with the consistency condition (68) and the bilinear ansatz (76), are sufficient to determine the exponents $e(\rho,\sigma,k,r,s\,|\,m,n)$. Surprisingly, one does not need the analyticity and non-zeroedness of $T_{pq}$ in the central physical domain $\mathscr{D}_0$, i.e., the relation (73).

The same is true for $N$ even and $\rho-\sigma-k$ odd. In both these cases we find that there is no explicit dependence of $e(\rho,\sigma,k,r,s\,|\,m,n)$ on the types $r$, $s$ of the Riemann sheet.

For $N$ even and $\rho-\sigma-k$ even they are not quite sufficient, but they leave only $[(N+4)/4]$ parameters to be determined. If one also (guided by the other cases) assumes that $e(\rho,\sigma,k,r,s\,|\,m,n)$ is not explicitly dependent on $r$ or $s$, then there is only one undetermined coefficient left.

As we remarked above, a knowledge of the orders of the poles and zeros of a meromorphic function does not by itself fix the function, but it goes a long way towards it. For instance, one might observe that the orders $\epsilon_{pq}(k)$ satisfy the relations (47) and then guess the full set of recursion relations (43)–(45), and hence (5)–(7). One could then test these relations from series expansions. They imply the vital Assumption 2 of ref. 6, so one could then use the method of ref. 6 to obtain the result (67) therein.

## 9. SUMMARY

We have shown that the partition function per site $T_{pq}$ lives on a Riemann surface with an infinite number of sheets, each sheet corresponding to a point on a $(2N-1)$-dimensional lattice. However, one should note that adjacent sheets need not correspond to any simple geometric definition of adjacent points: the adjacency rules are given by (12) and (14).

It is a meromorphic function on this surface, with zeros and poles only when $t_p = \omega^k t_q$ for $k = 0,..., N-1$. The orders of these zeros (the negative of the order of the poles) are bilinear in the integers $m_0,..., n_{N-1}$ that specify the sheet. If we assume this bilinearity (with coefficients that may depend on $k$ and the parities and types of the Riemann sheet), then for $N$ odd they can be obtained from the rotation and inversion relations. For $N$ even this procedure does not uniquely fix the orders, but does determine them to within a small number $[(N+4)/4]$ of free parameters.

The integers $m_0,..., n_{N-1}$ (more precisely the differences of the $m_j$, and of the $n_j$) are connected with the variables of the hyperelliptic parametrization of the chiral Potts model[9] (with $u$, $v$ therein interchanged). We hope to discuss this point in a later paper, and at least present an elliptic function expression for $\tau_2(t_q)$ in the case $N = 3$.

A significant motivation for this work has been the still outstanding problem of obtaining the spontaneous magnetization (order parameter) of the chiral Potts model. There is an elegant conjecture (Eq. (3.13) of ref. 15, Eq. (1.20) of ref. 16, Eq. (15) of ref. 17) for this property, which is almost certainly true, but has not been proved. Following the method of Jimbo, Miwa, and Nakayashiki,[18] the author has derived functional relations for a generalized order parameter.[19, 20] These have a similar structure to the inversion/rotation relations for the free energy. If one could solve them, then one would have verified the conjecture.

It was therefore the author's hope that the techniques of this paper could be applied to solving the order parameter relations. It must be admitted that preliminary results are not encouraging. If we assume that the function lives on the same Riemann surface as $T_{pq}$ (which is not obvious), and that the orders (exponents) of the poles and zeros when $t_p^N = t_q^N$ are bilinear in the $m_j, n_j$, then for $N$ odd we find the solution is unique, but is merely the "wrong solution" we obtained in Eq. (72) of ref. 19. So we are no further forward.

It is possible that this function has poles and zeros other than those when $t_p^N = t_q^N$: we have found some suggestion of this in preliminary low-temperature expansion calculations, and hope to discuss this in a later paper. If so, then the order parameter function may be much more complicated than $T_{pq}$, and the simple ideas we used here may need considerable expansion.

## ACKNOWLEDGMENTS

This work was supported in part by the Australian Research Council

## REFERENCES

1. R. J. Baxter, Free energy of the solvable chiral Potts model, *J. Stat. Phys.* **52**:639–667 (1988).
2. R. J. Baxter, *Calculation of the Eigenvalues of the Transfer Matrix of the Chiral Potts Model*, Proc. Fourth Asia-Pacific Physics Conference (Seoul, Korea, 1990), Vol. 1 (World-Scientific, Singapore, 1991), pp. 42–58.
3. R. J. Baxter, Chiral Potts model: Eigenvalues of the transfer matrix, *Phys. Lett. A* **146**:110–114 (1990).
4. R. J. Baxter, Equivalence of the two results for the free energy of the chiral Potts model, *J. Stat. Phys.* **98**:513–535 (2000).
5. H. Au-Yang, B.-Q. Jin, and J. H. H. Perk, Baxter's solution for the free energy of the chiral Potts model, *J. Stat. Phys.* **102**:471–499 (2001).
6. R. J. Baxter, The "inversion relation" method for obtaining the free energy of the chiral Potts model, cond-mat/0212075, to appear in *Phys. A*.
7. R. J. Baxter, The inversion relation method for some two-dimensional exactly solved models in lattice statistics, *J. Stat. Phys.* **28**:1–41 (1982).
8. R. J. Baxter, Elliptic parametrization for the three-state chiral Potts model, in *Integrable Quantum Field Theories*, L. Bonara *et al.*, eds. (Plenum Press, New York, 1993), pp. 27–37.
9. R. J. Baxter, *Hyperelliptic Function Parametrization for the Chiral Potts Model*, Proc. Intnl. Congress of Mathematicians, Kyoto, 1990 (Springer-Verlag, Tokyo, 1990), pp. 1305–1317.
10. R. J. Baxter, Elliptic parametrization of the three-state chiral Potts model, in *Integrable Quantum Field Theories*, L. Bonora *et al.*, eds. (Plenum Press, New York, 1993), pp. 27–37.
11. R. J. Baxter, Some hyperelliptic function identities that occur in the chiral Potts model, *J. Phys. A*. **31**:6807–6818 (1998).
12. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, New solutions of the star-triangle relations for the chiral Potts model, *Phys. Lett. A* **128**:138–142 (1988).
13. R. J. Baxter, V. V. Bazhanov, and J. H. H. Perk, Functional relations for transfer matrices of the chiral Potts model, *Internat. J. Modern Phys. B* **4**:807–870 (1990).
14. R. J. Baxter, Superintegrable chiral Potts model: Thermodynamic properties, an "inverse model," and a simple associated Hamiltonian, *J. Stat. Phys.* **57**:1–39 (1989).
15. S. Howes, L. P. Kadanoff, and M. den Nijs, Quantum model for commensurate-incommensurate transitions, *Nuclear Phys. B* **215[FS7]**:169–208 (1983).
16. G. Albertini, B. M. McCoy, J. H. H. Perk, and S. Tang, Excitation spectrum and order parameter for the integrable *N*-state chiral Potts model, *Nuclear Phys. B* **314**:741–763 (1989).
17. M. Henkel and J. Lacki, Integrable chiral $Z_n$ quantum chains and a new class of trigonometric sums, *Phys. Lett. A* **138**:105–109 (1989).

18. M. Jimbo, T. Miwa, and A. Nakayashiki, Difference equations for the correlation functions of the eight-vertex model, *J. Phys. A* **26**:2199–2210 (1993).
19. R. J. Baxter, Functional relations for the order parameters of the chiral Potts model, *J. Stat. Phys.* **91**:499–524 (1998).
20. R. J. Baxter, Functional relations for the order parameters of the chiral Potts model: low temperature expansions, *Phys. A* **260**:117–130 (1998).